EIGENVALUE PROBLEM – EXAMPLE

Consider the second-order tensor $T_{ij}$ whose components in a given basis are

$$T_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}. \quad (1)$$

Find its eigenvalues and corresponding left eigenvectors.

Because the tensor is symmetric ($T_{ij} = T_{ji}$), we expect the eigenvalues and eigenvectors to be real, and we know that the eigenvectors will be orthogonal.

**Step 1: Find the eigenvalues**

We first find the eigenvalues by solving the equation $\det(T_{ij} - \lambda \delta_{ij}) = 0$, or in matrix form:

$$\det \begin{bmatrix} 1 - \lambda & 0 & 0 \\ 0 & 1 - \lambda & 2 \\ 0 & 2 & 1 - \lambda \end{bmatrix} = 0. \quad (2)$$

Calculating the determinant,

$$(1 - \lambda)^3 - 4(1 - \lambda) = 0, \quad (3)$$

which can be factored as

$$(1 - \lambda)[(1 - \lambda)^2 - 4] = 0, \quad (4)$$

from which the three eigenvalues are found as $\lambda_1 = 1$, $\lambda_2 = -1$, and $\lambda_3 = 3$.

**Step 2: Find the corresponding eigenvectors**

Next, we solve for the eigenvectors one by one, by looking for a solution of the system $n_i (T_{ij} - \lambda \delta_{ij}) = 0$ for each of the three possible values of $\lambda$. Note that for each eigenvalue the system is overdetermined, i.e. there exists an infinite number of solutions for $n_i$: any non-zero solution is an eligible eigenvector. Typically, the eigenvectors are chosen to be of norm 1.

* Eigenvector corresponding to $\lambda_1$: It is obtained by solving

$$[n_1 \ n_2 \ n_3] \times \begin{bmatrix} 1 - \lambda_1 & 0 & 0 \\ 0 & 1 - \lambda_1 & 2 \\ 0 & 2 & 1 - \lambda_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \text{i.e. } \begin{cases} \quad 0 = 0 \\ 2n_3 = 0 \\ 2n_2 = 0 \end{cases} \quad (5)$$

We see that $n_2 = n_3 = 0$, and $n_1$ is arbitrary so we can pick any nonzero value. For instance: $n_{(1)} = [1 \ 0 \ 0]$ is an eigenvector with eigenvalue $\lambda_1 = 1$.

* Eigenvector corresponding to $\lambda_2$: Similarly,

$$[n_1 \ n_2 \ n_3] \times \begin{bmatrix} 1 - \lambda_2 & 0 & 0 \\ 0 & 1 - \lambda_2 & 2 \\ 0 & 2 & 1 - \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \text{i.e. } \begin{cases} \quad 2n_1 = 0 \\ 2n_2 + 2n_3 = 0 \\ 2n_2 + 3n_3 = 0 \end{cases} \quad (6)$$

We see that $n_1 = 0$, and from the second and third equations $n_2 = -n_3$. Let’s take $n_2 = 1$ and $n_3 = -1$. After normalization, the eigenvector is $n_{(2)} = [0 \ 1 \ -1]/\sqrt{2}$.

* Eigenvector corresponding to $\lambda_3$: Following the same procedure for $\lambda_3$, we find $n_{(3)} = [0 \ 1 \ 1]/\sqrt{2}$.

We note that $(n_{(1)}, n_{(2)}, n_{(3)})$ forms an orthonormal basis, as expected given that the tensor $T_{ij}$ is symmetric.