Chapter 4
Conservations Laws and Physical Properties of Fluids

Problem 4.1. Prove the following properties of the material derivative (where \( f \) and \( g \) are two field variables):

(a) \[
\frac{D}{Dt}(f + g) = \frac{Df}{Dt} + \frac{Dg}{Dt},
\]

(b) \[
\frac{D}{Dt}(fg) = f \frac{Dg}{Dt} + g \frac{Df}{Dt},
\]

(c) \[
\frac{D}{Dt}(f \circ g) = (f' \circ g) \frac{Dg}{Dt} \quad \text{(here, \( \circ \) denotes composition)}
\]

Problem 4.2. Let \( S \) be the surface of a deformable body in three dimensions, with outward unit normal \( n \), and let \( f(x, t) \) be some scalar field variable. Show that

\[
\frac{d}{dt} \int_S f \, n \, dS = \int_S \frac{\partial f}{\partial t} \, n \, dS + \int_S (w \cdot n) \nabla f \, dS,
\]

where \( w \) is the velocity on the surface of the body.

Problem 4.3. Consider a one-dimensional velocity field given by:

\[
v_x = v_x(x, t), \quad v_y = 0, \quad v_z = 0.
\]

The density in the flow is known and given by \( \rho(x, t) = \rho_0(2 - \cos \omega t) \) where \( \rho_0 \) and \( \omega \) are constants. Find an expression for \( v_x(x, t) \) if \( v_x(0, t) = U \).

Problem 4.4. Consider the flow of an incompressible fluid with constant density \( \rho \) through an axisymmetric nozzle of length \( L \) with varying cross-section. Let \( x \) be the coordinate along of the axis of the nozzle, and let \( r \) be the radial coordinate in cylindrical coordinates. Also denote by \( R(x) \) the local radius of the nozzle. Assume that the fluid velocity inside the nozzle can be expressed as

\[
v(x) = v_x(x)e_x - \alpha r e_r,
\]

where \( \alpha \) is a constant.

(a) Use the continuity equation to solve for \( v_x(x) \), with the condition \( v_x(0) = V_0 \) at the entrance of the nozzle.

(b) Knowing that the velocity vector along the nozzle wall must be tangential to the wall, obtain an expression for the shape \( R(x) \) of the nozzle in terms of \( \alpha, V_0, \) and the entrance radius \( R_0 = R(0) \). Realizing that the nozzle shape must be independent of \( V_0 \), explain why \( \alpha \) and \( V_0 \) must be proportional.

(b) Calculate the flow rates through the entrance and exit of the nozzle, located at \( x = 0 \) and \( x = L \), respectively. Prove that the two flow rates are equal. Is this result surprising?

Problem 4.5. Consider a control volume \( V \) in a fluid. Its bounding surface \( S \) has outward unit normal \( n_i \). The hydrodynamic torque exerted by the fluid outside of the control volume on the fluid inside is given in index notation by:
\[ L_i = \int_S \epsilon_{ijk} x_j T_{kl} n_l \, dS \]  
(4.4)

where \( T_{kl} \) denotes the total stress tensor.

(a) Rewrite equation (4.4) using Gibbs notation.

(b) Show using index notation that the hydrodynamic torque on the control volume \( V \) can also be expressed as

\[ L_i = \epsilon_{ijk} \int_V \left( T_{kj} + x_j \frac{\partial T_{km}}{\partial x_m} \right) \, dV. \]

(4.5)

**Problem 4.6.**

(a) Consider the differential equation for conservation of linear momentum derived in the lecture. After division by the density \( \rho \), take the curl of the equation to obtain a new equation for the fluid vorticity \( \omega = \nabla \times v \).

(b) Assume that the flow is two-dimensional and that the viscous stress tensor is zero: \( \tau = 0 \). Prove that if the body forces are conservative, meaning that there is a scalar potential \( \phi \) such that \( f = \nabla \phi \), then the vorticity simply moves with the fluid (i.e. \( \partial \omega / \partial t = 0 \)).

**Problem 4.7.**

Prove that, for any fluid property \( f \) and material control volume \( V(t) \), the following is satisfied:

\[ \frac{d}{dt} \int_{V(t)} \rho f \, dV = \int_{V(t)} \rho \frac{Df}{Dt} \, dV. \]

(4.6)

**Problem 4.8.**

Consider a still fluid where the density is variable and the total stress tensor is given by \( T_{ij} = -p \delta_{ij} \). Write down the momentum equation with a gravity field expressed as

\[ f_i = \frac{\partial \phi}{\partial x_i} \]

(4.7)

where \( \phi \) is the gravitational potential. Derive the relation \( \nabla \rho \times \nabla \phi = 0 \).

**Problem 4.9.**

A cart with frictionless wheels holds a water tank, motor, pump, and nozzle. The cart is on horizontal ground and initially still. At time zero, the cart has a mass \( M_0 \) and the pump is started to produce a jet of water with area \( A \), jet velocity \( U_{jet} \) (constant with respect to the cart) at \( 30^\circ \) to the horizontal. Find and solve the equations governing the mass and velocity of the cart as a function of time.

**Problem 4.10.**

Prove that a Newtonian fluid with constant viscosity in incompressible flow obeys the relation

\[ \nabla \cdot \tau = \mu \nabla^2 \mathbf{v}. \]

(4.8)

**Problem 4.11.**

For a Newtonian fluid that satisfies the Stokes assumption, show that the viscous dissipation, defined as \( \Phi = \tau : \nabla \mathbf{v} \), is given by

\[ \Phi = -\frac{2}{3} \mu (\nabla \cdot \mathbf{v})^2 + 2 \mu \mathbf{E} : \mathbf{E}. \]

(4.9)

**Problem 4.12.**

Consider the following velocity field defined in spherical coordinates \((r, \theta, \phi)\):

\[ v_r = U \cos \theta \left[ 1 - \left( \frac{a}{r} \right)^3 \right], \quad v_\theta = -U \sin \theta \left[ 1 + \frac{1}{2} \left( \frac{a}{r} \right)^3 \right], \quad v_\phi = 0. \]

(4.10)

(a) Is the flow compressible or incompressible?

(b) Calculate all nonzero components of the viscous stress tensor \( \tau \) as functions of \( r \) and \( \theta \).
(c) Calculate the viscous dissipation $\Phi$ (defined in Problem 4.11).
(d) Calculate the integral of $\Phi$ over the volume $a \leq r \leq \infty$.

**Problem 4.13.** A commonly used equation of state for water is approximately independent of the temperature:

$$\frac{P}{p_0} \approx (A + 1) \left( \frac{\rho}{\rho_0} \right)^n - A,$$  \hspace{1cm} (4.11)

where $A \approx 3000$, $n \approx 7$, $p_0 = 1$ atm, and $\rho_0 = 998 \text{ kg.m}^{-3}$. From this formula, determine:
(a) the pressure (in atm) required to double the density of water;
(b) the speed of sound in water at 1 atm.

**Problem 4.14.** The objective of this problem is to prove the following relation between the thermal equation of state $p = p(\rho, T)$ and the caloric equation of state $e = e(\rho, T)$:

$$\frac{\partial e}{\partial \rho} = \frac{1}{\rho^2} \left[ p - T \left( \frac{\partial p}{\partial T} \right)_\rho \right].$$  \hspace{1cm} (4.12)

(a) Using the first law of thermodynamics, in the form $de = Tds + (p/\rho^2)d\rho$, show that

$$ds = \frac{1}{T} \left\{ \left( \frac{\partial e}{\partial \rho} \right)_T \rho - \rho^2 \right\} d\rho + \left( \frac{\partial e}{\partial T} \right)_\rho dT.$$  \hspace{1cm} (4.13)

From this relation, obtain expression for $(\partial s/\partial \rho)_T$ and $(\partial s/\partial T)_\rho$.
(b) Using the result of part (a), and the fact that

$$\frac{\partial}{\partial T} \left( \frac{\partial s}{\partial \rho} \right)_T = \frac{\partial}{\partial \rho} \left( \frac{\partial s}{\partial T} \right)_\rho,$$  \hspace{1cm} (4.14)

prove equation (5).
(c) Using the results of parts (a) and (b), also show that

$$\left( \frac{\partial s}{\partial \rho} \right)_T = -\frac{1}{\rho^2} \left( \frac{\partial p}{\partial T} \right)_\rho.$$  \hspace{1cm} (4.15)