Chapter 3
Flow Kinematics

Problem 3.1. Consider the two-dimensional unsteady flow with velocity field \((v_1, v_2) = (2x_1 + t, x_2 - 2t)\). Find
(a) the streamline through the point \((1, 1)\) at \(t = 1\);
(b) the pathline for the fluid element which is at the point \((1, 1)\) at \(t = 1\);
(c) the streakline through the point \((1, 1)\) at \(t = 1\).
Plot all three curves on the same graph.

Problem 3.2. Consider the two-dimensional flow field defined by the following velocity components:
\[ u_1 = \frac{u_2}{1+t}, \quad u_2 = 1. \]  
(3.1)
For this flow field, find the equation of
(a) the streamline through the point \((1, 1)\) at \(t = 0\);
(b) the pathline for a particle released at the point \((1, 1)\) at \(t = 0\);
(c) the streakline at \(t = 0\) that passes through the point \((1, 1)\).

Problem 3.3. A two-dimensional flow field has the following velocity components:
\[ u_1 = x(1 + t), \quad u_2 = 1. \]  
(3.2)
Determine the following quantities for this flow field:
(a) the equation of the streamline that passes through the point \((1, 1)\) at time \(t = 0\);
(b) the equation of the pathline for a particle released at the point \((1, 1)\) at time \(t = 0\);
(c) the equation of the streakline that passes through the point \((1, 1)\) as seen at \(t = 0\).

Problem 3.4. In a Cartesian coordinate system \((x_1, x_2, x_3)\), a flow has the following velocity components:
\[ v_1 = \alpha x_2, \quad v_2 = \beta x_1^2, \quad v_3 = 0. \]  
(3.3)
(a) Is the flow compressible or incompressible?
(b) Find the equation of the streamline passing through the point \((x_1, x_2) = (1, 1)\).
(c) Calculate the rate-of-strain tensor \(E_{ij}\) and rotation tensor \(W_{ij}\). Determine the principal rates of extension.

Problem 3.5. Consider the two-dimensional flow field with velocity components in a Cartesian coordinate system given by:
\[ (u, v, w) = (2y, x, 0). \]  
(3.4)
(a) Determine the components of the acceleration field.
(b) Is the flow compressible or incompressible?

(c) Find the equation of the streamline passing through the point with coordinates \((3, 1, 0)\).

**Problem 3.6.** Sketch the streamlines for the following linear flows \(u_j(x) = x_i \Gamma_{ij}\) (where \(\Gamma_{ij} = \partial u_j / \partial x_i\) is the constant velocity gradient). Determine the principal axes and principal rates of extension of \(E_{ij} = \frac{1}{2} (\Gamma_{ij} + \Gamma_{ji})\) in each case.

\[
\begin{align*}
(a) \quad \Gamma_{ij} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, & (b) \quad \Gamma_{ij} &= \begin{bmatrix} 0 & 2 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, & (c) \quad \Gamma_{ij} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\
(d) \quad \Gamma_{ij} &= \begin{bmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}, & (e) \quad \Gamma_{ij} &= \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix},
\end{align*}
\]

**Problem 3.7.** Consider the sink flow defined by the vector field

\[
\mathbf{u}_i(x) = -\frac{\sigma x_i}{r^2}, \quad r^2 = x_k x_k,
\]

where \(\sigma > 0\) (with units of \(\text{length}^3/\text{time}\)) is a constant which determines the magnitude of the sink.

(a) Determine the local rate of strain tensor \(E_{ij}\) and rotation tensor \(W_{ij}\) at an arbitrary point in space. What are the principal axes and principal rates of extension for \(E_{ij}\)? Describe the local flow in a line or two.

(b) A common and simple model for a polymer is an elastic dumbbell (see figure below) which satisfies the following equation for its end-to-end vector \(\mathbf{R}_i\):

\[
\frac{d\mathbf{R}_i}{dt} - \mathbf{R}_j \frac{\partial u_i}{\partial x_j} + \frac{1}{2\lambda} \mathbf{R}_i = 0,
\]

where the constant \(\lambda\) is called the relaxation time of the dumbbell. The position \(x_i\) of the center of the dumbbell also satisfies the equation

\[
\frac{d\mathbf{x}_i}{dt} = u_i(x).
\]

Assume we place the dumbbell in the sink flow such that its orientation lies along the principal axis of maximum extension of \(E_{ij}\). If the dumbbell didn’t move (\(\frac{dx_i}{dt} = 0\)), would it stretch or contract? Does it matter where you put it?

(c) Now allow the dumbbell to move assuming that \(|\mathbf{R}_i|(t = 0) = R_0\) and \(|\mathbf{x}_i|(t = 0) = r_0\). Find the dumbbell’s position and stretch (i.e. \(|\mathbf{R}_i|\)) as a function of time. Determine the condition under which \(|\mathbf{R}_i|(t_{\text{final}}) > R_0\).
Problem 3.8. A spherical drop of colored dye is placed at the center of a two-dimensional flow field with velocity \( v_j(x) = x_i \Gamma_{ij} \) and follows the fluid. The uniform velocity gradient \( \Gamma_{ij} = \partial u_i / \partial x_j \) is given by:

\[
\Gamma_{ij} = \dot{\varepsilon} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix},
\]

where \( \dot{\varepsilon} > 0 \) is a constant parameter.

(a) Describe the flow field with words and explain qualitatively what you expect the streamlines to look like (no calculations needed).

(b) Explain qualitatively how the shape of the colored fluid will evolve with time (justify your answer).

(c) Determine quantitatively the evolution with time of the dimensions of the colored drop in the \( x, y, \) and \( z \) directions. Denote by \( a \) the initial radius of the drop.

Problem 3.9. If a trace (say a dye) of concentration \( c \) is introduced into a fluid flow and the dye does not diffuse, then \( c \) satisfies the conservation equation

\[
\frac{Dc}{Dt} = 0
\]

where \( D/Dt \) denotes the substantial derivative, which simply states that the dye follows the fluid. Assume the velocity field is given by

\[
u_i(x) = \frac{x_i}{r^2}, \quad r^2 = x_j x_j.
\]

(a) Sketch the streamlines of the flow. In which direction is the local fluid velocity?

(b) Now assume a thin cylindrical ring of dye (see figure above) of initial thickness \( b_0 \) and radius \( R_0 \) (with \( b_0 \ll R_0 \)) is placed in the flow centered at the origin. Describe the shape of the ring as it evolves a short time later. How does the shape change? How does the small dimension change with time? How does the large dimension change with time? Be as quantitative as possible.
**Problem 3.10.** In a table of vector differential operators, look up the expression for $\nabla \times \mathbf{v}$ in a cylindrical coordinate system.

(a) Compute the vorticity for the flow in a round tube of radius $R$ where the velocity profile is

$$v_r = 0, \quad v_\theta = 0, \quad v_z = v_0 \left[1 - \left(\frac{r}{R}\right)^2\right]. \quad (3.13)$$

(b) Compute the vorticity for an ideal vortex where the velocity is

$$v_r = 0, \quad v_\theta = \frac{\Gamma}{2\pi r}, \quad v_z = 0. \quad (3.14)$$

(c) Compute the vorticity in the vortex flow given by

$$v_r = 0, \quad v_\theta = \frac{\Gamma}{2\pi r} \left[1 - \exp \left(-\frac{r^2}{4\nu t}\right)\right], \quad v_z = 0. \quad (3.15)$$

Sketch all velocity profiles and vorticity profiles.

**Problem 3.11.** In cylindrical coordinates, compute the components of the strain rate tensor and vorticity tensor for the Burgers vortex, whose velocity components are

$$v_r = -ar, \quad v_\theta = \frac{\Gamma}{2\pi r} \left[1 - \exp \left(-\frac{r^2}{2\nu a}\right)\right], \quad v_z = 2az. \quad (3.16)$$

**Problem 3.12.** The “point vortex” flow in two dimensions has the velocity field

$$u = -ULy \frac{x^2}{x^2 + y^2}, \quad v = ULx \frac{y^2}{x^2 + y^2} \quad (\text{for } x^2 + y^2 \neq 0), \quad (3.17)$$

where $U, L$ are reference values of speed and length.

(a) Show that the flow can represented in Lagrangian coordinates as

$$x(a, b, t) = R_0 \cos(\omega t + \theta_0), \quad x(a, b, t) = R_0 \sin(\omega t + \theta_0), \quad (3.18)$$

where $R_0^2 = a^2 + b^2$, $\theta_0 = \arctan(b/a)$, and $\omega = UL/R_0^2$.

(b) Consider at $t = 0$ a small rectangle of marked fluid particles determined by the points $A(L, 0)$, $B(L + \Delta x, 0)$, $C(L + \Delta x, \Delta y)$, and $D(L, \Delta y)$. If the points move with the fluid, once point $A$ returns to its initial position what is the shape of the marked region? Since $(\Delta x, \Delta y)$ are small, you may assume the region remains a parallelogram.