MAE 210A – MIDTERM EXAMINATION

Wednesday November 5 2014

Open notes and handouts, closed book. No calculators, computers, cell phones.
Show all your work and clearly explain every step of your derivations.

PROBLEM 1: Hydrodynamic torque on a control volume (10 points)
Consider a control volume $V$ in a fluid, with bounding surface $S$ and outward unit normal $n_i$. The hydrodynamic torque exerted by the fluid outside of the control volume on the fluid inside is given in index notation by:

$$L_i = \int_S \epsilon_{ijk} x_j T_{kl} n_l \, dS$$

where $T_{kl}$ denotes the total stress tensor.
(a) Rewrite equation (1) using Gibbs notation.
(b) Show using index notation that the hydrodynamic torque on the control volume $V$ can also be expressed as

$$L_i = \epsilon_{ijk} \int_V \left( T_{kj} + x_j \frac{\partial T_{km}}{\partial x_m} \right) \, dV.$$  

PROBLEM 2: Acceleration field (10 points)
Let $\mathbf{v}(\mathbf{x}, t)$ be an Eulerian velocity field. Using index notation, show that the acceleration field $\mathbf{a}(\mathbf{x}, t)$ satisfies the relation

$$\mathbf{a}(\mathbf{x}, t) = \nabla \mathbf{v} + \nabla \left( \frac{1}{2} \mathbf{v} \cdot \mathbf{v} \right) + \mathbf{\omega} \times \mathbf{v},$$

where $\mathbf{\omega}$ is the vorticity vector. [Hint: simplify the term $\mathbf{\omega} \times \mathbf{v}$ using the definition of the vorticity]

PROBLEM 3: Hydrodynamic stretching of a DNA molecule (20 points)
The mapping of genes on a DNA molecule requires stretching the DNA, which can be thought of as a ball of entangled yarn. One technique for stretching DNA is proposed in the figure below and relies on hydrodynamic flow: one end of the DNA is attached to a wall at the intersection of two microfluidic channels, and water is flown in the direction shown by the arrows by imposing a pressure difference $p_1 - p_2 > 0$. If the flow is strong enough, the drag force exerted by the water on the molecule can cause it to unravel and stretch.

Near the center point $O$ where the DNA is attached, the velocity field $\mathbf{u}$ is steady and two-dimensional in the $(x_1, x_2)$ plane with components

$$u_1(x_1, x_2) = -\beta x_1 x_2, \quad u_2(x_1, x_2) = \frac{\beta}{2} x_2^2,$$

(4)
where \( \beta \) is a constant (with units of \( s^{-1} \cdot m^{-1} \)) that depends on the pressure difference, fluid properties, and channel geometry.

(a) Is the flow compressible or incompressible?

(b) Calculate the equation of the streamline passing through the point \((x_1^0, x_2^0)\).

(c) Determine the rate-of-strain tensor \( E_{ij} \), rate-of-rotation tensor \( W_{ij} \), and vorticity vector \( \omega_i \). What are the principal axes and rates of extension of the flow? Describe in detail what happens to the shape of fluid elements near the origin.

(d) A simple model for the DNA molecule consists of a bead and a spring, which can be stretched by the flow as a result of the viscous drag on the bead, but stays coiled when there is no flow, see figure below. The velocity \( \mathbf{V}(t) \) of the bead results from the drag force due to the fluid velocity \( \mathbf{u} \), and from the spring force, which is proportional to the end-to-end vector \( \mathbf{R}(t) \) of the molecule:

\[
\mathbf{V}(t) = \frac{d\mathbf{R}}{dt} = \mathbf{u}(\mathbf{R}) - k\mathbf{R}.
\] (5)

Using equation (5), write down the two equations satisfied by the components \( R_1(t) \) and \( R_2(t) \). By considering the sign of \( dR_1/dt \), show that the flow tends to align the molecule along the \( x_2 \)-axis.

(e) **(BONUS QUESTION: +3 points)** Assuming the bead is initially placed at position \((R_1, R_2) = (0, h)\) at \( t = 0 \), solve for its trajectory \( R_2(t) \). What is the condition on \( h \) for the molecule to be stretched by the flow?